Harmonic Path Analysis

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Abstract

The article proposes a conceptual framework for a special type of experiments in harmonic analysis and discusses aspects of its implementation in a software tool - called *HarmoRubette*. The framework comprises three basic components, namely (1) a *harmonic configuration space* $HARM$ equipped with harmonic tensor ht quantifying the transitions between harmonic loci, (2) a *Riemann Logics* RL, quantifying the signification of harmonic loci through chords and (3) a *best path calculation method* based on the assumption of a first-order-transition model.[1] Aside from systematic and implementatory aspects much attention is paid to the study of concrete examples: the space of Riemann functions according to the concept of the >classic< HarmoRubette, Elaine Chew's Spiral array model, Fred Lerdahl's hybrid chordal/regional space.

1 Paths in Harmonic Spaces

Musical harmony is a fascinating subject domain gaining a growing interest from researchers in several disciplines - music theorists, psychologists, neuro-physiologists, computer scientists, as well as mathematicians. However, this subject domain is far from being conceptually grasped. The various approaches do not just differ in their ways of interpreting or explaning commonly accepted facts, but they significantly differ in their understanding of what is the relevant data to be interpreted. The present paper investigates a special type of approach which may be labeled *first order transition models*. They describe analyses of chord sequences in terms of pathways through abstract harmonic spaces, where »first order« refers to the strong assumption—or restriction—, namely that pathways can be understood simply from the investigation of dyadic transitions. The interest in this study emerged from the authors' involvement in the extension of the software *RUBATO* (c.f. Garbers (2003b) in this volume as well as Mazzola (2002)) and was driven by the desire to make the analytical approach behind the *HarmoRubette*—designed by

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1 Mazzola (2002), chapter 27, and especially section 27.2, presents considerations on a more general framework of harmonic semantics into which the present one can be embedded as a special case.

Guerino Mazzola and collaborators—more transparent and fully controllable by the user. It is not our music-theoretical intention to naively argue in favor of first order transition models. But we do argue that experiments on the basis of such a simple framework may offer useful insights for researchers with different theoretical perspectives: to ›advocates‹ of first order transition models who may strengthen or relativize their arguments and to their critics or to the advocates of more sophisticated approaches who may try to interpret the results in the context of their theory.

The investigations presented in this article led to an extended experimental framework consisting of re-design of the basic *HarmoRubette* as well as a strategy allowing the user to integrate his/her own approaches.

1.1 Chords, Harmonic Loci, and Harmonic Analyses

A semiological point of view offers a suitable way to describe aspects of musical harmony in terms of a sign system the syntactic layer of which is constituted by a vocabulary $CHORDS$ of chords and the semantic layer of which is constituted by a harmonic configuration space $HARM$. In the present article we discuss several concrete specifications of these two layers in detail.

The search for principles according to which chord syntagms are associated with configurations of harmonic loci could be labeled as a »study of harmonic signification«. However, we restrict our concept of *harmonic signification* to the elementary case of context-free significations $X \triangleright H$ of harmonic loci H by isolated chords X. We do not insist in a two-valued logic behind these significations, i.e. instead of sharply considering a relation $\triangleright \subseteq CHORDS \times HARM$ with the associated characteristic function $\chi_\triangleright : CHORDS \times HARM \to \{0, 1\}$ we consider a »fuzzy« *Riemann Logic* $RL : CHORDS \times HARM \to \mathbb{R}$ attributing (generalized) truth values $RL(X \triangleright H)$ to the significations $X \triangleright H$.[2]

Besides our restriction to elementary significations, we consider only the simplest type of syntagms, namely *chord sequences* $\mathcal{S} = (X_k)_{k=1,\ldots,n}$ and associate them with sequences $\mathcal{S} \triangleright \mathcal{H} := (X_k \triangleright H_k)_{k=1,\ldots,n}$ of isolated significations of the chords in $\mathcal{S}$. Within our framework we call such a sequence a *harmonic analysis* of the chord sequence $\mathcal{S}$. Its pure semantic layer, i.e. sequence $\mathcal{H} = (H_k)_{k=1,\ldots,n}$ of harmonic loci is called the *harmonic path* of the analysis $\mathcal{S} \triangleright \mathcal{H}$. The term *transition* refers only to ordered pairs (H_k, H_{k+1}) of consecutive loci within the semantic layer. i.e. it does not include the underlying chords. We write $H_k \rightsquigarrow H_{k+1}$. Behind this decision there is another strong restriction of our framework: Chord successions are not studied directly (i.e. syntactically), but only indirectly in terms of transitions between signified harmonic loci. Hence, the data actually taken into considera-

2 There are several motivations behind the term ›Riemann Logic‹. Firstly, Hugo Riemann himself compared musical activity with logical reasoning. Secondly, does the harmonic ambiguity and especially in Riemannian functional harmony suggest a treatment in terms of fuzzy logic. Thirdly, does the attempt to apply set theory to music—as in the case of »American Set Theory«—naturally imply the critical question for the role of logics. Recall that sets are the semantic models for classical logics. In subsection 2.3 we review a link to Topos Logic applied to harmonic morphemes.

tion in a harmonic analysis $(X_i)_{i=1,\ldots,n} \triangleright (H_i)_{i=1,\ldots,n}$ can be suitably displayed as follows:

$$\begin{array}{ccccccccc} X_0 & & X_1 & & X_2 & & X_3 & & X_4 \\ \triangledown & & \triangledown & & \triangledown & & \triangledown & & \triangledown \\ H_0 & \rightsquigarrow & H_1 & \rightsquigarrow & H_2 & \rightsquigarrow & H_3 & \rightsquigarrow & H_4 \end{array}$$

Remark 1 *With regard to the Viterbi algorithm (see subsection 1.4) it is easy to extend the framework to the study of proper dyadic significations* $(X_{k-1} \rightsquigarrow X_k) \triangleright (H_{k-1} \rightsquigarrow H_k)$*, see remark 3. Music-theoretically, this would allow to include the study of counterpoint and voice leading. However, we exclude these aspects from the investigations of this paper.*

1.2 Interpretation of Harmonic Path Analyses and Refinements

There is no intended >automatic< music-theoretical interpretation of the »best analyses« $\mathcal{S} \triangleright \mathcal{H}$. The formal meaning of »best« in the sense of the following subsections is not meant normatively. Practically, the path evaluation can be done—and interpreted—with a varying analytical scope: It can be applied globally at once to an entire given chord sequence $\mathcal{S}$, but it can also be applied locally to suitable subsequences $\mathcal{S}'$ of $\mathcal{S}$. And in the latter case one may further investigate the behavior of local analyses at overlaps, i.e. one may ask whether local best paths can be glued together coherently. As an extreme case of local analysis one may extract a sequence of *local germs*[3]. In this case the user choses a local window size by specifying the *causal* and *final depth*, i.e. the numbers of chords to be considered ahead of and after X_k. The resulting analytical germ sequence $(H_1, ..., H_n)$ consists of harmonic loci, each of which H_k is signified by X_k in a locally best path, i.e. within the analysis window corresponding to X_k. However, the isolated H_k in the germ-sequence do not carry anymore this contextual information.

Remark 2 *The analytical results of the HarmoRubette 1 are germ sequences. In the HarmoRubette 2 one obtains best-path analyses (c.f. subsection 1.4). In order to experiment with germ sequences it is recommended to calculate small window-size analyses in batch processing and to extract the germs out of them. This can be done in the* OHR-*framework (see Garbers (2003a) as well as Garbers (2003b) in this volume). Besides of this, the original implementation is available as well.*

1.3 Quantification of Harmonic Transition

Suppose we are about to quantitatively measure harmonic transitions $H_1 \rightsquigarrow H_2$. Two strategies are opposed to one another: Either, higher values may express a higher amount of necessary >effort< to realize a transition, or, higher values are just meant to directly >evaluate< a transition. We formalize these two possibilities

3 In sheaf theory »germs« are equivalence classes of functions which share the same local behaviour in a given point p. To verify that two functions represent the same germ, one has to find a suitable small neighbourhood around p where they coinside

differently. In the first case we allow non-negative values as well as negative ones. Positive values express >inhibitions< for the corresponding transitions, while nagative values express >attractions<. In the second case we allow non-negative values only. We use the following notation and terminology:

1. A map $ht : HARM \times HARM \to \mathbb{R}$ is called a *harmonic tensor*. In this case we have in mind that its values quantify transitions in terms of inhibitions (non-negative values) and attractions (negative values). We speak of *para-pseudo-distance*, if all ht is symmetric (i.e. $ht(H_1 \rightsquigarrow H_2) = ht(H_2 \rightsquigarrow H_1)$, for all loci H_1, H_2) and positive semi-definite, i.e. $ht(H_1 \rightsquigarrow H_2) \geqslant 0$ and $ht(H \rightsquigarrow H) = 0$ for all loci H_1, H_2, H. A para-pseudo-distance is called a *pseudo-distance* if further the triangle-inequality $ht(H_1 \rightsquigarrow H_2) + ht(H_2 \rightsquigarrow H_3) \geqslant ht(H_1 \rightsquigarrow H_3)$ is satisfied for all $H_1, H_2, H_3 \in HARM$. It is called a *para-distance*, if it is positive definite (i.e. $ht(H_1 \rightsquigarrow H_2) = 0$ if and only if $H_1 = H_2$). Finally, ht is called a *distance* (or a *metric*), if it is a pseudo-distance and a para-distance.
2. A map $HT : HARM \times HARM \to [0, \infty)$ is called a *harmonic transition value map*. In this case we have in mind that its values directly quantify transitions in a monotonous way. If the image of HT is actually $[0, 1]$ we call it a *para-probability map*. If further $\sum_{H \in HARM} HT(H_1, H) = 1$ for all $H_1 \in HARM$ we speak of a *semi-probability-map* and if in addition we have $\sum_{H \in HARM} HT(H, H_1) = 1$ for all $H_1 \in HARM$ we speak of a *probability-map*.

We use the exponential/logarithmic functions in order to formally translate the two kinds of quantification into one another: Suppose, we are given a harmonic tensor $ht : HARM \times HARM \to \mathbb{R}$. Its associated harmonic transition value map $HT = e^{-ht}$ is defined by

$$HT(H_1 \rightsquigarrow H_2) = exp(-ht(H_1 \rightsquigarrow H_2)).$$

Conversely, if we are given a harmonic transition value map $HT : HARM \times HARM \to [0, \infty)$, then its associated transition value map $ht = -log(HT)$ is defined by

$$ht(H_1 \rightsquigarrow H_2) = -log(HT(H_1 \rightsquigarrow H_2)).$$

Non-negative harmonic tensors (including para-pseudo-distances) formally correspond to para-probability-maps. However, we do not intend to project an ontological interpretation onto this correspondence.

1.4 Evaluation of Harmonic Analyses

We now discuss a numeric evaluation method for harmonic analyses $\mathcal{S} \triangleright \mathcal{H}$ as well a suitable algorithm for the determination of best analyses for a fixed chord sequence $\mathcal{S}$ and varying pathways $\mathcal{H}$. The algorithm is called *Viterbi algorithm* and is used in the context of *Hidden Markov Models* in order to calculate a most probable process in accordance with a sequence of observations. Readers which are familiar with such models will notice that such a probabilistic interpretation can be seen as special case of our framework. The chord sequence would then play the role of the observations being made and the pathway in the harmonic space would be interpreted as a hidden stochastic process. However, the Viterbi algorithm is just based on the assumption that the evaluation of pathways can be obtained step by step in terms of an order-preserving evaluation of partial pathways. This assumption does not presuppose a stochastic interpretation.

Consider a chord sequence $\mathcal{S} = (X_k)_{k=0,\ldots,n}$ and associated harmonic paths $\mathcal{H} = (H_k)_{k=0,\ldots,n}$ to be evaluated as candidates for best harmonic analyses $\mathcal{S} \triangleright \mathcal{H}$. We sketch a very general situation in which the Viterbi algorithm works. For each index $k = 1, ..., n$ we consider

1. a function $transVal_k : [0, \infty) \times HARM \times HARM \to [0, \infty)$ evaluating path-continuing transitions $(v, H_{k-1} \rightsquigarrow H_k)$ which depend only on the value v of the previous path and the two loci H_{k-1} and H_k of that transition,

2. a function $locusVal_k : [0, \infty) \times HARM \to [0, \infty)$ evaluating path-specific choices (v, H_k) which depend only on the values v of the previous paths (leading to H_k) and the concrete choice of H_k at index k, which of course includes dependence upon the chord X_k at index k.

Remark 3 *Both functions $transVal_k$ and $locusVal_k$ may depend upon k, i.e. they may depend upon the chord sequence $\mathcal{S}$. Within our framework $locusVal_k$ in fact substantially depends upon k, because it encodes the significations $X_k \triangleright H_k$, but $transVal_k$ does not depend upon k. The investigation of proper dyadic significations would require the definition of variable functions $transVal_k$.*

As an essential presupposition we need that $transVal_k$ and $locusVal_k$ are both *order-preserving* in their first argument, whenever they do not vanish. Further we assume them to be *zero-preserving*. This latter condition just means that zero values stand for discarded transitions or loci which should not occur at all in any analysis, i.e. $transVal_k(0, H_{k-1} \rightsquigarrow H_k) = 0$ and $locusVal_k(0, H_k) = 0$ for all $H_{k-1}, H_k \in HARM$. According to the first condition, does $v_1 < v_2$ imply

$$\begin{array}{lll} transVal_k(v_1, H_{k-1} \rightsquigarrow H_k) & < & transVal_k(v_2, H_{k-1} \rightsquigarrow H_k), \\ & & \text{whenever} \quad transVal_k(v_2, H_{k-1} \rightsquigarrow H_k) > 0 \\ locusVal_k(v_1, H_k) & < & locusVal_k(v_2, H_k), \\ & & \text{whenever} \quad locusVal_k(v_2, H_k) > 0 \end{array}$$

If we further consider an initial evaluation $eval_0 : HARM \to [0, \infty)$ we define the values $eval_k((H_0, ..., H_k))$ of the increasing partial paths of $\mathcal{H}$ as:

$$\begin{array}{ll} eval_1((H_0, H_1)) & = locusVal_1(\\ & \quad transVal_1(eval_0(H_0), H_0 \rightsquigarrow H_1), H_1) \\ & \ldots \\ eval_k((H_0, ..., H_k)) & = locusVal_k(\\ & \quad transVal_k(eval_{k-1}(H_0, ..., H_{k-1}), H_{k-1} \rightsquigarrow H_k), H_k) \end{array}$$

Note, that the partial evaluation maps are order-preserving too. The total value of a path $\mathcal{H} = (H_0, ..., H_n)$ is its last partial value, i.e. $eval(\mathcal{H}) = eval_n(\mathcal{H})$. A *best* *path* is a path $\mathcal{H}^*$ with a maximal value, i.e. $eval(\mathcal{H}^*) \geqslant eval(\mathcal{H})$ for all paths $\mathcal{H}$ of the same length. Best paths have the property that all their partial sub-paths are best sub-paths too. This is implied by the property of order-preservation. The Viterbi algorithm for best path calculation is based on this fact and works like this: For each $H_1 \in HARM$ one runs through all $H_0 \in HARM$, calculates $eval_1((H_0, H_1))$ and stores the maximal value $maxVal_1(H_1)$ among these as well as the list $Predecessors_1(H_1)$ of all those H_0 for which the maximal value $maxVal_1(H_1) = eval_1((H_0, H_1))$ is obtained. According to the order preservation of $locusVal_1$ the maximum $maxVal_1(H_1)$ equals

$$locusVal_1(\max_{H_0 \in HARM}(transVal_1(eval_0(H_0), H_0 \rightsquigarrow H_1), H_1))),$$

i.e. $locusVal_1$ has to applied just once, namely to the maximum of the transition values towards H_1 which saves calculation time. Suppose now that we have already calculated $maxVal_{k-1}(H_{k-1})$ as well as $Predecessors_{k-1}(H_{k-1})$ for all $H_{k-1} \in HARM$. At index k we fix each element $H_k \in HARM$, run trough all $H_{k-1} \in HARM$, and similarly calculate

$$maxVal_k(H_k) = locusVal_k(\\ \max_{H_{k-1} \in HARM}(transVal_k(maxVal_{k-1}(H_{k-1}), H_{k-1} \rightsquigarrow H_k), H_k))),$$

an collect those precedessors H_{k-1} which actually yield this maximal value in the set $Predecessors_k(H_k)$.

Best Paths are obtained backwards, starting from a locus $H_n \in HARM$ for which $maxVal_n(H_n) > 0$ is maximal compared to all other $H'_n \in HARM$. Given such a best final locus H_n one selects a locus H_{n-1} from $Predecessors_n(H_n)$, a locus H_{n-2} from $Predecessors_{n-1}(H_{n-1})$ and so forth until a full path is selected backwards. It is obvious from the above construction that one obtains all best paths by browsing through the implied graph of possible predecessors in the described way.

1.5 Formulas for $transVal$ and $locusVal$

In the present subsection we provide specific formulas for $locusVal_k$ and $transVal_k$ in accordance with the concrete music-theoretical examples to be discussed in the following sections. Suppose we are given

1. a *Riemann Logic* $RL : CHORDS \times HARM \to [0, \infty)$ and
2. a *Harmonic Transition Value Map* $HT : HARM \times HARM \to [0, \infty)$

A positive constant $c > 0$ regulates the relative influence of the transition values against the locus values. In the unmarked case we choose $c = 1$, while $c > 1$ gives higher weight to transitions and $c < 1$ gives higher weight to local significations.

Further we suppose that we are given a chord sequence $\mathcal{S} = (X_k)_{k=0,\dots,n}$ and finally, a sequence of *custom restriction maps* $\rho_k : HARM \to [0, 1], \quad (k = 0, ..., n)$.

These maps are useful for computer-aided explorative analyses, as we will see below. With these settings we define:

$$\begin{array}{ll} transVal_k : & [0,\infty) \times HARM \times HARM \to [0,\infty) \\ locusVal_k : & [0,\infty) \times HARM \to [0,\infty) \\ eval_0 : & HARM \to [0,\infty) \end{array}$$

by the formulas

$$\begin{array}{ll} transVal_k(v, H_{k-1}, H_k) & := v \cdot HT(H_{k-1} \rightsquigarrow H_k), \\ locusVal_k(v, H_k) & := v \cdot c \cdot RL(X_k \triangleright H_k) \cdot \rho_k(H_k). \\ eval_0(H_0) & := locusVal_0(1, H_0) = c \cdot RL(X_0 \triangleright H_0) \cdot \rho_0(H_0). \end{array}$$

Multiplication with positive numbers is order-preserving as well as zero-preserving. Multiplication with zero is also allowed and corresponds to the disqualification of those paths passing through the loci (or transitions) with zero-evaluations at k. These formulas completely specify the data which is necessary for the best path calculation. In case of a harmonic tensor $ht : HARM \times HARM \to \mathbb{R}$ we consider the corresponding maps $transVal_k(v, H_{k-1}, H_k) := v \cdot exp(-ht(H_{k-1} \rightsquigarrow H_k))$ (c.f. subsection 1.3).

In the normal case we have constant restriction maps $\rho_k(H_k) = 1$, which actually are not restrictive. But in explorative applications one is sometimes interested to force the pathways to pass though a certain locus Hat index k. In this case one uses a restriction map of the kind

$$\rho_k(H_k) = \begin{cases} 1 & for \quad H_k = H \\ 0 & else \end{cases}$$

forcing any path $\mathcal{H}$ to pass trough H at position k.

2 Riemann Logics

In this section we discuss some specifications of the very general setting, namely to define a map

$$RL : CHORDS \times HARM \to [0,\infty).$$

In this general situation chords are simply elements of the abstract set $CHORDS$, i.e. they are not necessarily composed of tones or intervals.

2.1 General Morphology of Chords

Chord Morphology—in a broad sense—is the internal investigation of a chord vocabulary $CHORDS$ in preparation of the study of harmonic signification. A special—but music-theoretically central—case is the study of chords as *tone sets*. This implies the consideration of a tone space $TONES$ with

$$CHORDS = Fin(TONES) \subseteq 2^{TONES},$$

i.e. in this case chords are understood as the finite subsets of the space $TONES$. Another possibility (to be discussed in the subsequent subsection) models chords as *tone profiles* or ›fuzzy tone sets‹.

In Hugo Riemann's terms the semantic layer of harmony is consituted by *tonal functions* within given keys and modes (see Riemann (1887) as well as sections 3, 4). These are signified first of all by *prime chords*, i.e. major or minor triads. To each key, mode and function there is a »prototypical« prime chord signifying that locus (e.g. the C-major-triad signifying the C-major tonic). Other chords signifying the same locus (e.g. the A-minor triad) are studied in *morphological* relation to the prototypical one. (The A-Minor-triad shares *consonant* tones C and E with the prime chord and has an additional *dissonant* tones, namely A being ›conceptually dissonant‹ with respect to the C-major-triad). Dissonant tones supporting the signification are called *characteristic* dissonances.

Therefore, a ›Riemann-inspired‹ strategy to define a Riemann Logic RL is to start with a map $locusChord : HARM \to CHORDS$, and to first associate each harmonic locus $H \in HARM$ with a »prototypical« chord $X = locusChord(H)$, which yields the highest truth value $RL(X \triangleright H)$. In a second step one attempts to determine $RL(X' \triangleright H)$ for other chords X' by *comparing* them with X on a morphological level. In other words, one may define

$$RL(X \triangleright H) := compare(X, locusChord(H))$$

on the basis of the two maps

$$\begin{aligned} locusChord: &\quad HARM \to CHORDS, \\ compare: &\quad CHORDS \times CHORDS \to [0, \infty), \end{aligned}$$

To directly edit the individual values $RL(X \triangleright H)$ for every tone set and every locus H ›by hand‹ is practically impossible. Even in the simple case of a 12-elemented tone set and 72 harmonic loci there are $(2^{12} - 1) \cdot 72 = 294840$ values to be specified. Of course, one may object that most of these values seem counter-factual with respect to a selected corpus of musical works. But even for the actually used chords one needs either statistical methods or a computational model. The latter involve formal components which extrapolate a Riemann Logic from a small selection of parameters.

2.2 Tone Profiles

In this subsection we discuss a linear extrapolation approach which is based on calculations with *tone profiles*. Our goal is to specify the map *compare*. First consider the real vector space $\mathbb{R}^{TONES}$ freely generated over the set $TONES$. This space has an independent one-dimesional subspace for each tone $t \in TONES$. Tone profiles are defined as vectors

$$(x_t)_{t \in TONES} \in \mathbb{R}^{TONES}$$

with the following properties:

- $x_t \geqslant 0$ for all $t \in TONES$ and $x_t > 0$ for only finitely many t,

- $\sum_{t \in TONES} x_t^2 = 1$.

The set of all tone profiles (with respect to $TONES$) is denoted by $Pro(TONES)$. Geometrically speaking, tone profiles are points on the positive quadrant of the unit sphere in $\mathbb{R}^{TONES}$ with finitely many non-vanishing coordinates. Now we introduce a pair of maps

$$Pro(TONES) \underset{charChord}{\overset{toneSet}{\rightleftarrows}} Fin(TONES)$$

such that tone sets can be considered as special tone profiles, or—more precisely—such that the composition $toneSet \circ charChord$ yields the identity on $Fin(TONES)$ (in other words: $Fin(TONES)$ becomes a *retract* in $Pro(TONES)$). These maps are defined as follows: The *toneSet* map sends each tone profile $X = (X_t)_{t \in TONES}$ to its carrier set

$$toneSet(X) = |X| := \{t \in TONES \mid x_i \neq 0\},$$

i.e. to the finite set of those tones t for which the coordinates x_t do not vanish. The *charChord* map sends each finite tone set $T \subset TONES$ to its normalized characteristic function $\chi_T : TONES \to [0, 1]$ with

$$\chi_T(t) = \begin{cases} card(T)^{-2} & \text{for} \quad t \in T \\ 0 & \text{otherwise.} \end{cases}$$

which can be interpreted as a tone profile.

In this setup it is useful to simply set $CHORDS = Pro(Tones)$ and to define a *comparison*-map for tone profiles. A natural solution is the canonical scalar product

$$\langle ., . \rangle : \mathbb{R}^{TONES} \times \mathbb{R}^{TONES} \to \mathbb{R}$$

according to which the vector basis $TONES$ becomes an *orthonomal* base. If one restricts this scalar product to tone profiles $X, Y \in Pro(TONES)$, $\langle X, Y \rangle$ yields always values between 0 and 1 namely the cosinus of the angle between X and Y. In addition one may concatenate the scalar product of profiles with a suitably chosen monotone function $f : [0, 1] \to [0, \infty)$. Hence, we define the comparison map as follows:

$$pro_f = f \circ \langle ., . \rangle : Pro(TONES) \times Pro(TONES) \to [0, \infty).$$

A monotone function f would not change the essential quality of the resulting Riemann Logic, but in connection with the best path calculation it is nevertheless a sensitive ingredience. In particular, the deformation f may be non-linear. Recall from our considerations in subsection 2.1, that—in order to complete the definition of the Riemann Logic RL—we need a map

$$locusProfile : HARM \to Pro(Tones)$$

attributing a tone profile to each harmonic locus. Such a map can practically be edited by hand, especially if one makes further homogeneity assumptions on the structure of tonalities. The *HarmoRubette* offers a suitable user-interface this (see section 3). Such maps $locusProfile$ can also be drawn from empirical data, where profiles represent statistical results like listeners judgments or relative occurrence rates of tones (c.f. Purwins (2003) in this volume).

2.3 Perspectival Morpho-Logic

There is a straight forward approach to calculate a $locusProfile$-map on the basis of the internal logics of *harmonic morphemes*. We admit, that a proper understanding of this >inserted< subsection requires familiarity with content and notation of Noll and Brand (2003) in this volume (especially subsections 3.1, 3.2 and 4.2). In this concrete situation we are concerned with the homogeneous 12-tone system $TONES = \mathbb{T} = \{0, 1, ..., 11\}$ and we associate harmonic loci with monoids $M \in \mathcal{MON}$ of (affine) tone perspectives $f = {}^b a : \mathbb{T} \to \mathbb{T}$ with ${}^b a(t) = at + b \, mod \, 12$. More precisely, to each monoid M we have the set Ω_M of its *left ideals* (or cosieves), which represent the truth values for (equivariant) characteristic functions in the internal logics of the M-actions on the chords $X \in Ext(M)$ in the extension $Ext(M)$ of M. In close analogy to profiles $p \in Pro(\mathbb{T})$ we have the characteristic functions $\chi_{|M|} : \mathbb{T} \to \Omega_M$. All we have to do, is to >downboil< the left ideals $B \in \Omega_M$ into numbers (as >fuzzy truth values<). We may choose any map $\lambda : \Omega_M \to [0, \infty)$ preserving the partial ordering within Ω_M, i.e. $B_1 \subset B_2$ must imply $\lambda(B_1) < \lambda(B_2)$, like the cardinality map $\lambda(B) := \sharp(B)$. Now, suppose we can associate each locus $H \in HARM$ with such a pair $\big(M(H), \lambda_H : \Omega_{M(H)} \to [0, \infty)\big)$, then we define the map

$$locusProfile : HARM \to \mathbb{R}^{\mathbb{T}} \quad \text{with} \quad locusProfile(H)[t]; = \lambda_h(\chi_{|H(M)|}(t)).$$

Finally, normalizing the vectors $(locusProfile(H)[t])_{t=0,...,11}$ leads to proper tone profiles in $Pro(\mathbb{T})$. As we know from Noll and Brand (2003) (section 4.2) tonal functions in Riemann's sense are closely related to the *bigeneric morphemes*

$$\mathcal{M}_{m,n} = (Int(Ext({}^m 3, {}^n 8), Ext({}^m 3, {}^n 8)) \quad \text{with} \quad 5m + 2n = \pm 1.$$

A purely morphological approach can thus just start with the 24-elemented space $HARM = \{(M_{m,n}, \lambda_{m,n}) \,|\, 5m + 2n = \pm 1\}$, (or the enlarged 48-elemented space including also *dissonant* morphemes with $5m + 2n = \pm 2\}$. The map $\lambda_{m,n}$ can be simply the cardinality map or any user-defined partial-order-preserving map.

We mention also another—closely related—approach, which is based on the setting $TONES = \mathbb{A} = \big\{{}^b a \,|\, a, b \in \{0, ..., 11\}\big\}$, i.e. where the 144 tone perspectives replace the role of >ordinary tones< $t \in \mathbb{T}$. Consequently we replace each >ordinary chord< $X \subseteq \mathbb{T}$ by the monoid $\mathbb{A}(X) = \{f \in \mathbb{A} \,|\, f(X) \subseteq X\}$ and use the map $charChord : \mathbb{A} \to Pro(\mathbb{A})$ to associate profiles to them. Once we are given a map $locusProfile : HARM \to Pro(\mathbb{A})$ we obtain a Riemann Logic $RL : Pro(\mathbb{A}) \times HARM \to [0, \infty)$ as described in the previous subsection. Thus we are done by just identifying the collection $HARM$ of harmonic loci with a suitable subset of $Pro(\mathbb{A})$. For any collection of monoids $M \subset \mathbb{A}$ like $HARM = \{(M_{m,n}, \lambda_{m,n}) \,|\, 5m + 2n =$

$\pm 1\}$ we can use the identification via $charChord$. Likewise we may refine these homogeneous profiles by attributing individual weights to the tone perspectives $f \in M_{m,n}$ and thus experimentally shaping $locusProfile(M_{m,n})$ for each harmonic locus.

Remark 4 *This second approach was implemented in the original* HarmoRubette *on the basis of Noll (1997) and is still avilable as the »Noll-Classic«-method. The Riemannian tonal functions are associated with profiles* $locusProfile(M_{m,n})$ *and can be edited in the »Noll-preferences«-panel. In a future study we intend to embed these methods into harmonic pathway analyses using mathematically »natural« harmonic tensors, like* Hausdorff-Metrics *on* $2^{\mathbb{A}}$ *associated with suitable metrics on* $\mathbb{A}$.

2.4 Concrete Tone Spaces

We close this section by defining some elementary tone spaces and derived structures in preparation of the subsequent sections. We start with *chromatic pitch height* $\mathbb{H} \simeq \mathbb{Z}$ encoded according to MIDI, where 60 represents the >middle C<. Further one has *octave identification* map

$$oct : \mathbb{H} \to \mathbb{H}_{oct} \cong \mathbb{Z}_{12} \quad \text{with} \quad oct(H) := H\, mod\, 12.$$

Next, we consider *note names* of the kind

$$N_a \quad \text{with} \quad N \in \{F, C, G, D, A, E, B\} \quad \text{and} \quad a \in \{..., \flat\flat, \flat, \emptyset, \sharp, \sharp\sharp, ...\}.$$

We indentify note names with integers by arranging them along the *line of fifths* $\mathbb{F} \cong \mathbb{Z}$ (i.e. $..., B_\flat, F, C, G, D, A, E, B, F_\sharp, ...$). Concretely we identify F with -1, C with 0, G with 1, etc.). Furthermore we consider the enharmonic and the diatonic projections, sending note names to their *enharmonic classes* and to their *diatonic classes*:

$$\begin{array}{lll} enh : & \mathbb{F} \to \mathbb{F}_{enh} \cong \mathbb{Z}_{12} & enh(k) := k \bmod 12 \\ dia : & \mathbb{F} \to \mathbb{F}_{dia} \cong \mathbb{Z}_{7} & enh(k) := k \bmod 7 \end{array}$$

The two different music-theoretical interpretations $\mathbb{H}$ and $\mathbb{F}$ of $\mathbb{Z}$, as well as two interpretations $\mathbb{H}_{oct}$ and $\mathbb{F}_{enh}$ of $\mathbb{Z}_{12}$ form the following diagram:

$$\begin{array}{ccc} \mathbb{F} & & \mathbb{H} \\ \downarrow enh & & oct \downarrow \\ \mathbb{F}_{enh} & \overset{\cdot 7\, mod\, 12}{\leftrightarrows} & \mathbb{H}_{oct} \end{array}$$

Remark 5 *The simultaneous consideration of all four tone spaces along these maps is a special instance of a* well formed tone system *(c.f Carey and Clampitt (1989)). Formally, such a simultaneous view of a* note name - pitch height concordance *can be described as the* limit *of the above diagram (consisting of those pairs* $(k, H) \in \mathbb{F} \times \mathbb{H}$ *of note names* k *and pitch heights* H *for which* $7 \cdot enh(k)\; mod\, 12 = oct(H)$ *holds) and fits into the general language of* forms *and* denotators *in the sense of Mazzola (2002) (chapter 6).*

We consider the natural—translation invariant—distance between integers on the line of fifths $d : \mathbb{F} \times \mathbb{F} \to \mathbb{N} \cup \{0\}$ with $d(x, y) = |x - y|$. Further we consider the induced enharmonic and the diatonic distance between note names, by virtue of:

$$d_{enh} : \quad \mathbb{F} \times \mathbb{F} \to \{0, 1, ..., 6\} \quad d_{enh}(x, y) := min\{d(x, z) \,|\, enh(y) = enh(z)\}$$
$$d_{dia} : \quad \mathbb{F} \times \mathbb{F} \to \{0, 1, 2, 3\} \quad d_{dia}(x, y) := min\{d(x, z) \,|\, dia(y) = dia(z)\}$$

The following definitions introduce elementary music-theoretical objects (c.f. sections 4 and 5)

- The sequence $Dia(k) := k + (-1, 0, 1, 2, 3, 4, 5) \subset \mathbb{F}$ of seven consecutive elements of the fifth line beginning from the note $k - 1$ is called *diatonic collection* with reference k.
- An *alteration* of $Dia(k)$ is any sequence $D = (n_{-1}, ..., n_5)$ of note names such that $dia(n_i) = dia(k + i)$ for $i = -1, ..., 5$. The *signature* of this alteration is the sequence $Sig(D) = \frac{1}{7}(D - Dia(k))$. We write $D = Dia(k)_{Sig(D)}$.
- The tripel $\tau_{Maj}(k) = (k, k + 1, k + 4)$ is called *major triad* with base note k.
- The tripel $\tau_{min}(k) = (k, k + 1, k - 3)$ is called *minor triad* with base note k.
- The tripel $\tau_{dim}(k) = (k, k - 6, k - 3)$ is called *diminished triad* with base note k.

3 *HarmoRubette* and Re-Design

The *HarmoRubette* was implemented as a plug-In of the *RUBATO*-software. The original software was created by Guerino Mazzola and Oliver Zahorka for the operating system NEXTSTEP and has been ported to Mac OSX and further extended by the second author.[4] The main concern of this paper is the harmonic path analysis for chord-sequences. We skip several practical aspects such as the translation of a score or its parts into a chord-sequence, the interpretation and usage of harmonic weights in performance experiments and refer to Fleischer (2003) and the RUBATO-Documentation. In this section we recapitulate the original approach and motivate the current extensions in the re-design of this tool.

3.1 Harmonic Configuration Space in the >Classic< *HarmoRubette*

The original version of the *HarmoRubette* implements a single 72-elemented space $HARM = \mathcal{R}$ of tonal functions with respect to the 12 enharmonic classes as keys. It is a cartesian product

$$\mathcal{R} = \mathbb{F}_{enh} \times \{T, D, S\} \times \{Maj, min\}$$

4 Further information on this OpenSource-Project see http://www. rubato.org.

of a twelve-elemented set of tonalities $\mathbb{F}_{enh}$, a three-elemented set $\{T, D, S\}$ of tonal functions and a two elemented set $\{Maj, min\}$ of modes. The harmonic tensor $ht : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$ is the sum of three user-defined tensors within these factors:

$$\begin{aligned} h_{ton}: &\quad \mathbb{F}_{enh} \times \mathbb{F}_{enh} \to \mathbb{R} \\ h_{fun}: &\quad \{T, D, S\} \times \{T, D, S\} \to \mathbb{R} \\ h_{mod}: &\quad \{Maj, min\} \times \{Maj, min\} \to \mathbb{R} \end{aligned}$$

The tonality tensor h_{ton} is supposed to be homogeneous with respect to the translation within $\mathbb{F}_{enh}$, i.e. it satisfies $h_{ton}(x, y) = h_{ton}(x+z, y+z)$ for all $x, y, z \in \mathbb{F}_{enh}$. The full harmonic tensor ht is defined by the formula:

$$h((t_1, f_1, m_1) \rightsquigarrow (t_2, f_2, m_2) := h_t(0 \rightsquigarrow (t_2 - t_1)) + h_f(f_1 \rightsquigarrow f_2) + h_m(m_1 \rightsquigarrow m_2).$$

The user has to specify $25 = 12 + 9 + 4$ values in the theory settings. The corre-

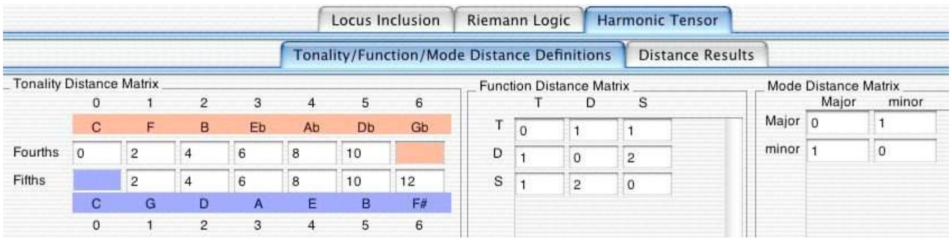


Figure 1: Snapshot of the user interface for controlling the harmonic tensor *ht*. It consists of three parts: tonality distance matrix (left), function distance matrix (center) and mode distance matrix (right)

sponding panels (c.f. figure 1) are called *tonality distance matrix*, *function distance matrix* and *mode distance matrix*. The harmonic tensor *ht* is a pseudometric or a metric if and only if all the three tensors h_{ton}, h_{fun} and h_{mod} are.

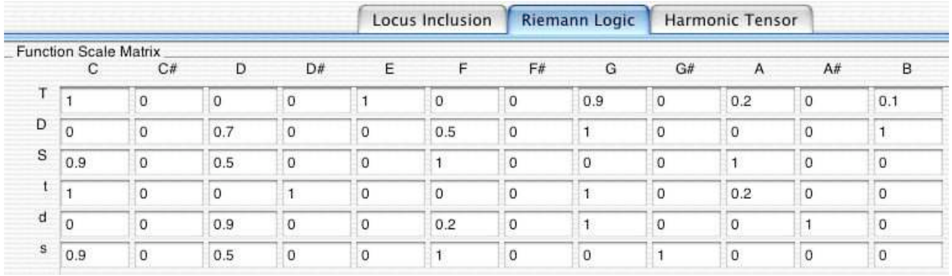
3.2 Riemann Logic in the >Classic< *HarmoRubette*

We first describe a slightly simplified definition in direct application of the tone profile method (c.f. subsection 2.2)and recall Mazzola's original account afterwards. We are concerned with the space $TONES = \mathbb{H}_{oct}$ of pitch classes and the corresponding tone profiles $CHORDS = Pro(\mathbb{H}_{oct})$. The definition of a map $locusProfile : \mathcal{R} \to Pro(\mathbb{H}_{oct})$ is given from a user defined map:

$$FunctionScale : \{0\} \times \{T, D, S\} \times \{Maj, min\} \to \mathbb{R}_{\geqslant 0},$$

and a homogenity assumption that the 6 profiles are shaped in the same way in all 12 tonalities. In other words, the user specifies the (not yet normalized) profiles for the 6 tonal functions with respect to the C-tonality. Figure 2 shows the corresponding user interface, which is called the *Function Scale Matrix*.
In accordance with the homogenity assumption we obtain the (normalized) formula:

$$locusProfile(k, f, m)(h) := \frac{FunctionScale(0, f, m)(h - 7k \bmod 12)}{\sqrt{\sum_{t=0}^{11} FunctionScale(0, f, m)(t)^2}}$$



	C	C#	D	D#	E	F	F#	G	G#	A	A#	B
T	1	0	0	0	1	0	0	0.9	0	0.2	0	0.1
D	0	0	0.7	0	0	0.5	0	1	0	0	0	1
S	0.9	0	0.5	0	0	1	0	0	0	1	0	0
t	1	0	0	1	0	0	0	1	0	0.2	0	0
d	0	0	0.9	0	0	0.2	0	1	0	0	1	0
s	0.9	0	0.5	0	0	1	0	0	1	0	0	0

Figure 2: Function Scale Matrix. The user has to specify six tone profiles for the tonal functions with respect to the C tonality.

and hence obtain the following simplified >classic< Riemann Logic:

$$RL_{direct}(X \triangleright H) := prof_f(X, locusProfile(H))$$

with the monotonous function

$$f(t) := \begin{cases} exp(t) & t \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The user has now access to the monotonous function f and may also experiment with the identity, the normal exponential function or a custom function.

In Mazzola's original implementation the formula for $RL_{direct}(X \triangleright H)$ was *not directly* applied to each given chord X but only to (realisations of) third chains. A *third-chain* is a sequence $K = (x_1, x_2, ..., x_k)$ with $x_i \in \{3, 4\} \subset \mathbb{Z}_{12}$ such that the corresponding sequence of partial sums $\Sigma(K) = (x_1, x_1 + x_2, \ldots, x_1 + x_2 + \ldots + x_k)$ contains no dublicates. The realisation of a third chain K at pitch class $x_0 \in H_{oct}$ is the tone set $R_{x_0}(K) := |x_0 + \sigma(K)|$. For any tone set X we consider the collection $TC(X)$ of all minimal third chain realisations containing X, namely

$$TC(X) := \{R_{x_0}(K) \,|\, X \subset R_{x_0}(K), length(K)\, minimal\}.$$

The present version of the *HarmoRubette* offers two ways of taking the (n-elemented) set $TC(X)$ into account (the second of which is the original proposal), namely

- to accumulate an average profile $p = c \cdot \sum_{Y \in TC(X)} charChord(Y)$ and to calculate $RL_{ThirdChain}(X \triangleright H) := RL_{direct}(p \triangleright H)$ (c = normalizing factor),
- to average the results of separately applying RL_{direct} to all characteristic profiles for minimal third chains

$$RL_{Mazzola-Classic}(X \triangleright H) := \frac{1}{n} \sum_{Y \in TC(X)} RL_{direct}(charChord(Y), H).$$

3.3 Aspects of the Re-Design

A major drawback of the original implementation was the unclear distribution of scientific responsibility between the creators of the tool and the user. On the one hand there was a high amount of pre-defined settings (such as the overall approach, the elements of space $HARM$, parametrized formulas, etc.). But on the other hand there was a high amount of user-defined parameters, which—from the user perspective—actually accessed a >black box<. Therefore, the guiding idea of our re-design of the program was to keep the overall idea of harmonic pathway analyses in the restricted sense to first-order-transition models, but—within this framework—to delegate the full scientific responsibility to the user, by giving full access to *all* parts of the analysis. Much effort has been invested by the second author to allow various ways of customization of the program at runtime. These new possibilities exemplify a general concept of software-integration in computer-aided experiments in music theory and analysis (see Garbers (2003b) in this volume, as well as Garbers (2003a)). With help of the scripting language $FScript$ the user may change the theory settings for a harmonic pathway analysis and may experiment with a harmonic space, Riemann Logics and best path calculation of his/her choice. [5] Another integration aspect is the possibility of remote control of the program for batch processing and other purposes. An *OpenMusic*-user [6] can integrate >wrapper-patches< into his/her own *Open-Music* programs and send chord-sequences as well as theory settings to $RUBATO$ and receive the results as Lisp-expressions. At the same time it is possible to inspect the results of these calculations (being stored in $RUBATO$) within the graphical user interface of the *HarmoRubette*. The second author also developed an *OpenMusic* library supporting the visual programming of *FScripts*.

A further practical drawback of the original tool was the high calculation time due to a purely combinatoric implementation for the best-path-selection. The Viterbi-algorithm offers a significant improvement. Furthermore the user may interactively force pathways to pass trough custom local selections (weighted filters) of harmonic loci (c.f. subsection 1.5).

Finally, there are recent publications offering proposals for harmonic configurations spaces and as well as for harmonic path analyses. These research activities added further motivation for this software project. In the subsequent sections 4 and 5 we recapitulate and reformulate the approaches of Chew (2000) and Lehrdahl (2001) by embedding them into the present framework.

4 Elaine Chew's Spiral Array

Elaine Chew (c.f. Chew (2000)) studies a hybrid space, which comprizes three levels of description, namely *tones*, *triads* and *keys* and distinguishes five families of music-theoretical objects each of which is parametrized by the abstract family of note names $k \in \mathbb{F}$.

5 After satisfactory experiments an expert can of course join the OpenSource project and implement the specific approach in ObjectiveC and thereby accelerate the calculations.

6 see for example Agon (2003) in this volume.

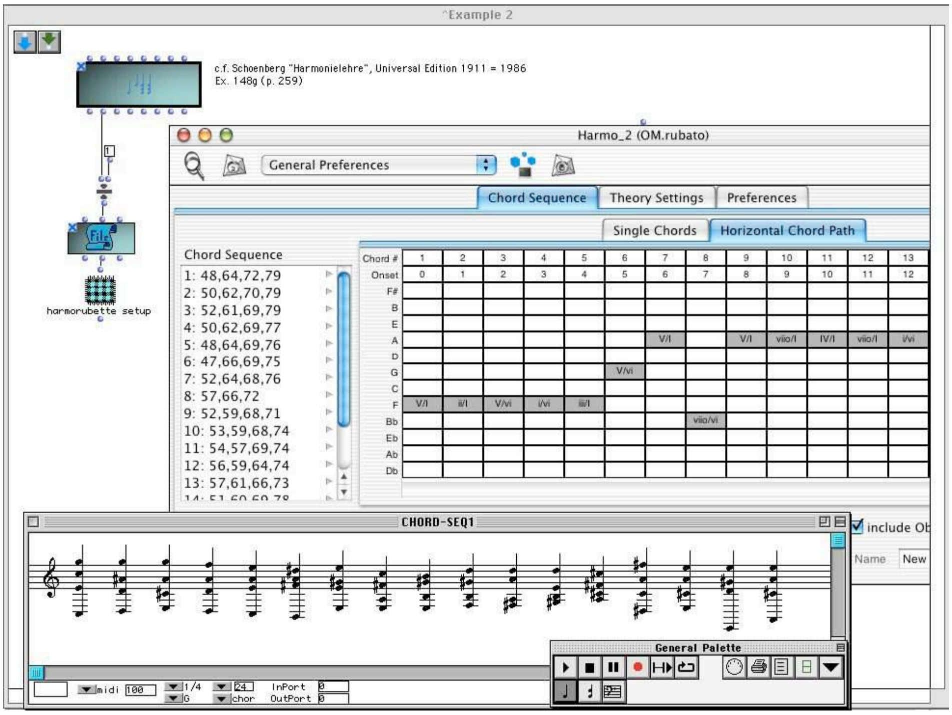


Figure 3: Snapshot of an integrated *OpenMusic-RUBATO* session: A chord sequence and theory-setting (Lerdahl's hybrid Space) are being configured in *OpenMusic* and the analysis is triggered by evaluating the »HarmoRubette-Setup«-Patch. The >best< pathway is shown in the *horizontal chord-path*-view of the *HarmoRubette*.

- On the level of *tones*[7] there corresponds a tone $\pi(k)$ to each note name $k \in \mathbb{F}$ (c.f. subsection 2.4).
- On the level of *triads* there corresponds a *major triad* $\tau_{Maj}(k)$ and a *minor triad* $\tau_{min}(k)$ to each note name $k \in \mathbb{F}$
- On the level of *keys* there corresponds a *major key* $\kappa_{Maj}(k)$ and *minor key* $\kappa_{min}(k)$ to each note name $k \in \mathbb{F}$. These are encoded as lists of triads

$$\begin{aligned}\kappa_{Maj}(k) &= (\tau_{Maj}(k), \tau_{Maj}(k+1), \tau_{Maj}(k-1)) \\ \kappa_{min}(k) &= (\tau_{min}(k), \tau_{Maj}(k+1), \tau_{min}(k+1), \tau_{min}(k-1), \tau_{Maj}(k-1))\end{aligned}$$

In this section re-interpet Chew's definitions in terms of harmonic configuration space and propose natural candidates for a Riemann Logic RL.

7 Chew speaks of 'pitches' instead of tones or note names. We keep close to Chew's notation but avoid the term *pitch*.

4.1 Tone-, Chord- and Keyspirals

The central idea in Chew's approach is to attribute suitable points to all these objects within one and the same three-dimensional Euclidean ambient space $\mathbb{R}^3$ in such a way that (1) the distances between these points are music-theoretically meaningful and that (2) transpositions are represented by Euclidean isometries (screw transformations). This is done by attributing suitable individual spirals to each of the five types, on which the corresponding discrete point sets are located. Chew defines five *representation maps*:

$$P, C_{Maj}, C_{min}, T_{Maj}, T_{min} : \mathbb{Z} \to \mathbb{R}^3$$

in the following way (c.f. Chew (2000) p. 59)

1. *tone representation*

$$P(k) = (r \cdot sin(\frac{k\pi}{2})r \cdot cos(\frac{k\pi}{2}), k \cdot h),$$

where radius r and height h are free parameters.

2. *major triad representation*

$$C_{Maj}(k) = w_1 \cdot P(k) + w_2 \cdot P(k+1) + w_3 \cdot P(k+4),$$

where $w = (w_1, w_2, w_3)$ is a weight vector of positive real numbers satisfying $w_1 + w_2 + w_3 = 1$. The point $C_{Maj}(k) \in \mathbb{R}^3$ is the 'center of effect' of the three weighted points $P(\tau_{Maj}(k))$ with respect to the weight vector w. It this situated inside of the triangle spanned by the corresponding pitch points (c.f. figure 4).

3. *minor triad representation*

$$C_{min}(k) = u_1 \cdot P(k) + u_2 \cdot P(k+1) + u_3 \cdot P(k-3),$$

where $u = (u_1, u_2, u_3)$ is a weight vector of positive real numbers satisfying $u_1 + u_2 + u_3 = 1$. The point $C_{min}(k) \in \mathbb{R}^3$ is the 'center of effect' of the three weighted points $P(\tau_{min}(k))$ with respect to the weight vector u.

4. *major key representation*

$$T_{Maj}(k) = \omega_1 \cdot C_{Maj}(k) + \omega_2 \cdot C_{Maj}(k+1) + \omega_3 \cdot C_{Maj}(k-1),$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ is a weight vector of positive real numbers satisfying $\omega_1 + \omega_2 + \omega_3 = 1$. The point $T_{Maj}(k) \in \mathbb{R}^3$ is the 'center of effect' of the three weighted triad representations of the triad collection $\kappa_{Maj}(k)$ with respect to the weight vector ω. It this situated inside of the triangle spanned by the corresponding triad centers (c.f. figure 4).

5. *minor key representation*

$$\begin{aligned} T_{min}(k) &= \nu_1 \cdot C_{min}(k) \\ &+ \nu_2 \cdot (\alpha \cdot C_{Maj}(k+1) + (1-\alpha) \cdot C_{min}(k+1)) \\ &+ \nu_3 \cdot (\beta \cdot C_{min}(k-1) + (1-\beta) \cdot C_{Maj}(k-1)), \end{aligned}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is a weight vector of positive real numbers satisfying $\nu_1 + \nu_2 + \nu_3 = 1$ and $0 \leqslant \alpha, \beta \leqslant 1$. The point $T_{min}(k) \in \mathbb{R}^3$ is the 'center of effect' of the five weighted triad representations of the triad collection $\kappa_{min}(k)$ with respect to the weight vector ν and the mode preferences α, β. The weight α controlles the relative influence of the major and the minor dominants to the center of effect of the minor key. Similarly, β controlles the relative influence of the minor and the major subdominants.

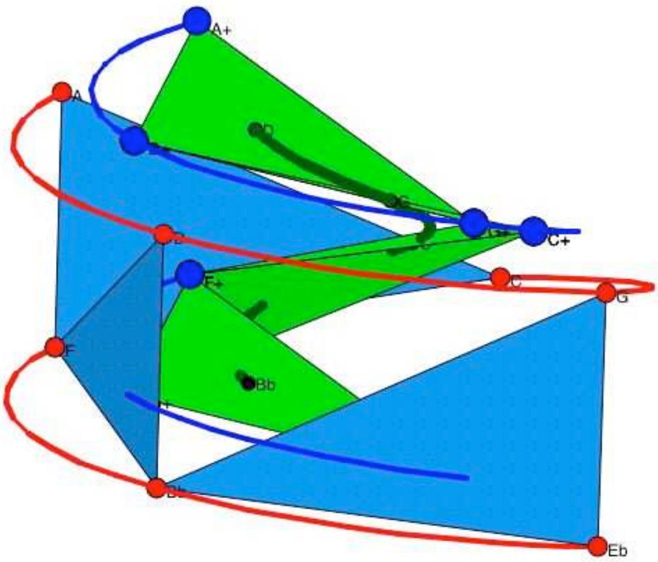


Figure 4: The figure shows the three spirals corresponding to the tones (outer spiral), major triads (middle spiral) and major keys (inner spiral)

4.2 Tuning of the Free Parameters

So far the spiral array representation is defined up to 16 non-negative real variables:

radius and height	(r, h)	
internal chord weights	$w = (w_1, w_2, w_3)$	$u = (u_1, u_2, u_3)$
internal key weights	$\omega = (w_1, w_2, w_3)$	$\nu = (u_1, u_2, u_3)$
mode preferences	(α, β)	

Chew (2000) (pp. 61 - 97) spends considerable effort in order to tune these variables with respect to music-theoretical constraints. A first such constraint deals with the total order of musical selected interval types[8], namely

$$\{P4, P5\} < \{M3, m6\} < \{m3, M6\} < \{M2, m7\} < \{m2, M7\} < \{d5, A4\}$$

If the ratio $a = \frac{h^2}{r^2}$ is chosen such that $\frac{2}{15} < a < \frac{2}{7}$, the Euclidean distances of corresponding pairs of tone representations exemplify the same total order, namely

$$\sqrt{2r^2 + h^2} < h < \sqrt{2r^2 + 9h^2} < 2\sqrt{r^2 + h^2} < \sqrt{2r^2 + 25h^2} < \sqrt{4r^2 + 36h^2}$$

Another set of constraints is concerned with the order of the Euclidean distances of the representations of triad tones and 'foreign' tones from those of the triads themselves:

$$\begin{array}{lcll} d(P(0), C_{Maj}(0)) & < & d(P(1), C_{Maj}(0)) & < \quad d(P(4), C_{Maj}(0)) \\ & < & d(P(l), C_{Maj}(0)) & \quad\;\; \forall l \neq 0, 1, 4. \end{array}$$

$$\begin{array}{lcll} d(P(0), C_{min}(0)) & < & d(P(1), C_{min}(0)) & < \quad d(P(-3), C_{min}(0)) \\ & < & d(P(l), C_{min}(0)) & \quad\;\; \forall l \neq 0, 1, -3. \end{array}$$

These conditions are satisfied, if

- $\frac{2}{15} < a < \frac{3}{15}$
- $3 < 4w_1 + 3w_2 < \frac{1}{5a} + \frac{5}{2}$
- $3 < 4w_1 + 3w_2 < \frac{1}{2a} + \frac{1}{2}$
- $\frac{1}{3}(u_2 + 1) < u_1$
- $(\frac{1}{8a} + \frac{3}{4}) \cdot u_1 + u_2 < \frac{1}{8a} + \frac{1}{4}$

Chew studies the border case $a = \frac{2}{15}$ in some detail, i.e. where fifths and major thirds are represented with the same Euclidean distance. In that case the above conditions restrict to

$$3 < 4w_1 + 3w_2 < 4, \quad u_2 + 1 < 3u_1, \quad 27u_1 + 16u_2 < 19.$$

A third type of constraints deals with the distances between tones and keys as well es between prominent intervals and keys. In particular on wants the following conditions to be satisfied for all $k \neq 0$ with respect to both modes $m \in \{Maj, min\}$:

pitch - key: $\frac{d(P(0),T_{Maj}(0))}{d(P(0),T_m(k))} < 1$

leading tone - key: $\frac{d(P(0),T_{Maj}(0))+d(P(5),T_{Maj}(0))}{d(P(0),T_m(k))+d(P(5),T_m(k)} < 1$

perfect fourth - key: $\frac{d(P(0),T_{Maj}(0))+d(P(5),T_{Maj}(0))}{d(P(0),T_m(k))+d(P(5),T_m(k)} < 1$

8 P4 = perfect fourth, P5 = perfect fifth, M3 = major third, m6 = minor sixth, ..., d5 = diminished fifth, A4 = augmented fourth.

The following set of parameters has been selected as an optimal solution to all these contraints (c.f. Chew (2000), pp. 94 - 97):

radius and height	$(1, \sqrt{\frac{2}{15}})$
internal chord weights	$w = (0.6025, 0.2930, 0, 1045)$
	$u = (0.6011, 0.2121, 0.1868)$
internal key weights	$\omega = \nu = w = (0.6025, 0.2930, 0, 1045)$
mode preferences	$(\frac{3}{4}, \frac{3}{4})$

Chew's empirical applications of this tuned model to *key finding* can be reinterpreted as a concrete case of the general framework presented here. The harmonic configuration space is then the space of the key-representations

$$\mathcal{C}_{key} = \{T_{Maj}(k) \,|\, k \in \mathbb{Z}\} \cup \{T_{min}(k) \,|\, k \in \mathbb{Z}\}$$

and the harmonic tensor is given by the Euclidean distance. We see in subsection 4.4 below, how the 'Center of Effect Generation' can be viewed as a special kind of Riemann Logic.

4.3 Proposal for a Riemann Function Space

We complete this review of Chew's spiral array model by proposing a harmonic configuration space $\mathcal{C}_{func} \subset \mathbb{R}^3$ of tonal functions, which are represented by 'centers of effect' on the line between the key representation and the chord representation of the prototypical triad associated with that tonal function. A user defined *function parameter* $0 < t < 1$ regulates the relative influence of the key center with respect to the chord center:

$$\begin{aligned}
T(k) &:= t \cdot T_{Maj}(k) &+(1-t) \cdot C_{Maj}(k)\\
D(k) &:= t \cdot T_{Maj}(k) &+(1-t) \cdot C_{Maj}(k+1)\\
S(k) &:= t \cdot T_{Maj}(k) &+(1-t) \cdot C_{Maj}(k-1)
\end{aligned}$$

$$\begin{aligned}
t(k) &:= t \cdot T_{min}(k) &+(1-t) \cdot C_{min}(k)\\
d_{\#}(k) &:= t \cdot T_{min}(k) &+(1+t) \cdot C_{Maj}(k+1)\\
d(k) &:= t \cdot T_{min}(k) &+(1-t) \cdot C_{min}(k+1)\\
s(k) &:= t \cdot T_{min}(k) &+(1-t) \cdot C_{min}(k-1)\\
s_{\#}(k) &:= t \cdot T_{min}(k) &+(1-t) \cdot C_{Maj}(k-1)
\end{aligned}$$

The configuration space

$$\mathcal{C}_{func} = \bigcup_{k \in \mathbb{Z}} \{T(k), D(k), S(k), t(k), d_{\#}(k), d(k), s(k), s_{\#}(k)\}$$

equipped with the Euclidean distance in $\mathbb{R}^3$ is a straight forward generalization of $\mathcal{C}_{key}$. For applications in romantic music one may also include the functions

$$\begin{aligned}
D_{\flat}(k) &:= t \cdot T_{Maj}(k) &+(1-t) \cdot C_{min}(k+1)\\
S_{\flat}(k) &:= t \cdot T_{Maj}(k) &+(1-t) \cdot C_{min}(k-1)
\end{aligned}$$

which would imply a modification of the major key representation into

$$\begin{aligned} T_{Maj}(k) &= \omega_1 \cdot C_{Maj}(k) \\ &+ \omega_2 \cdot (\alpha' \cdot C_{Maj}(k+1) + (1-\alpha') \cdot C_{min}(k+1)) \\ &+ \omega_3 \cdot (\beta' \cdot C_{Maj}(k-1) + (1-\beta') \cdot C_{min}(k-1)). \end{aligned}$$

4.4 Center-Of-Effect-Generator

Recall from subsection 4 that we are dealing with the tone space $TONES = \mathbb{F}$ of note names. Furthermore we choose the space of tone profiles $CHORDS = Pro(\mathbb{F})$ in order to define a Riemann-Logic. Recall also that in the spiral array model three levels of description are merged together. Especially chords are represented as 'centers of effect' of weighted tone sets. Hence, instead of interpreting harmonic loci within the space of tone profiles we can study the representations of tone profiles within the spiral array model. Let K denote the carrier of a tone profile $X = \sum_{k \in K} a_i \cdot k$. The representation $CE(X) := \sum_{k \in K} a_i \cdot P(k)$ is a point within the convex closure of the point set $P(K)$ in the euclidean space $\mathbb{R}^3$. Chew sucessfully applied this *Center of Effect* generation map CE to problems of key finding as well as to chord root finding(c.f. Chew (2000), pp. 99 - 138 as well as 158 respectively). It is natural to also try it in the determination of harmonic loci within the space $\mathcal{C}_{func} \subset \mathbb{R}^3$.
We propose the following Riemann Logics both of which translate high distances into small values near 0 and small distances into values near 1:

$$RL(X \triangleright H)_{inv} := \frac{1}{1 + \|CE(X) - H\|} \qquad RL(X \triangleright H)_{exp} := exp(-\|CE(X) - H\|).$$

Remark 6 *In her applications Chew (2000) obtains the coefficients* $0 < a_i \leqslant 1$ *within tone profiles* $X = \sum_{k \in K} a_i \dot{k}$ *from score data. Especially, she encodes tone durations and proposes also to consider metric weights. This is a general method which has to be used in connection with the other approaches too.*

5 Fred Lerdahl's model of Chordal/Regional Space

The present section reflects upon selected aspects from Fred Lerdahl's study of harmonic pathways in a harmonic configuration space which Lerdahl calls the *chordal/regional* space. Our investigations do not address Lerdahl's theoretical framework (c.f. Lehrdahl (2001)) as a whole but concentrate on his attempt to combine a *principle of hierarchy* with a *principle of shortest path* and we focus on the theoretical and practical problems which arise from this attempt for the definition of distances between harmonic loci. The following subsections present elementary investigations into Lerdahl's approach.

5.1 Intra-Regional Distance

According to Lerdahl's hierarchical viewpoint each tonal region - as an isolated space - is constituted by seven chord loci, namely the triadic degrees $Maj = \{I, ii, iii, IV, V, vi, vii^o\}$ in major regions and $Min = \{i, ii^o, III, iv, V, VI, \#vii^o\}$ in minor regions. He considers the metrics $\delta_{Maj} : Maj \times Maj \rightarrow [0, \infty)$ and $\delta_{min} : Min \times Min \rightarrow [0, \infty)$ which are given by the two tables below:

Table 1: Distance-Matrix of a Major Region

δ_{Maj}	I	ii	iii	IV	V	vi	vii^o
I	0	8	7	5	5	7	8
ii	8	0	8	7	5	5	7
iii	7	8	0	8	7	5	5
IV	5	7	8	0	8	7	5
V	5	5	7	8	0	8	7
vi	7	5	5	7	8	0	8
vii^o	8	7	5	5	7	8	0

Table 2: Distance-Matrix of a Minor Region

δ_{Min}	i	ii^o	III	iv	V	VI	$\#vii^o$
i	0	8	7	5	6	7	9
ii^o	8	0	8	7	6	5	8
III	7	8	0	8	9	5	8
iv	5	7	8	0	9	7	6
V	6	6	9	9	0	9	7
VI	7	5	5	7	9	0	9
$\#vii^o$	9	8	8	6	7	9	0

In subsection 5.6 we recapitulate Lerdahl's motivation of these concrete quantities in terms of *diatonic triad stratifications*. Meanwhile we consider them as arbitrary settings. In the next two subsections we are concerned with the study of 24 such regions. The regions of each mode are parametrized by the circle of fifths ($\mathbb{F}_{enh}$). Therefore we specify them by tones of reference or names, i.e. we write Maj($\mathbf{0}$) = Maj($\mathbf{C}$), Maj($\mathbf{1}$) = Maj($\mathbf{G}$), etc. as well as Min($\mathbf{0}$) = Min($\mathbf{c}$), Min($\mathbf{1}$) = Min($\mathbf{g}$), etc.

5.2 Inter-Regional Distance

Apart from these regions as separate spaces we consider an abstract *regional space Reg*, i.e. a space whose 24 elements are abstract *region loci*. To specify them by tones of reference or - preferably - by names we write

$$\mathbf{0}_+ = \mathbf{C},\ \mathbf{1}_+ = \mathbf{G}, ..., \ \mathbf{11}_+ = \mathbf{F} \quad \text{and} \quad \mathbf{0}_- = \mathbf{c},\ \mathbf{1}_- = \mathbf{g}, ..., \ \mathbf{11}_- = \mathbf{f}.$$

To introduce a metric on region loci we start by considering them as nodes of a *Kinship Graph* Γ_{Weber} representing *direct* regional kinship. This Graph $\Gamma_{Weber} = (Reg, Kin)$ consists of 48 edges besides its 24 nodes:

$$\begin{array}{rcl} Kin & = & \{(\mathbf{C},\mathbf{G}),(\mathbf{G},\mathbf{D}),...,(\mathbf{B_b},\mathbf{F}),(\mathbf{F},\mathbf{C})\} \\ & \sqcup & \{\mathbf{c},\mathbf{g}),(\mathbf{g},\mathbf{d}),...,(\mathbf{b_b},\mathbf{f}),(\mathbf{f},\mathbf{c})\} \\ & \sqcup & \{(\mathbf{C},\mathbf{a}),(\mathbf{G},\mathbf{e}),...,(\mathbf{B_b},\mathbf{g}),(\mathbf{F},\mathbf{d})\} \\ & \sqcup & \{(\mathbf{C},\mathbf{c}),(\mathbf{G},\mathbf{g}),...,(\mathbf{B_b},\mathbf{b_b}),(\mathbf{F},\mathbf{f})\} \end{array}$$

These 48 edges represent music-theoretically different types of direct (or first order) regional kinship, namely *fifth kinship* among Major regions and among minor regions as well as *relative kinship* and *parallel kinship* between Major and minor regions. The abstract graph Γ_{Weber} does not distinguish between these types. The concrete directions of edges in Figure 5 have no mathematical meaning. But note that only 37 edges out of the 48 are drawn. Me mention that the complete graph can be drawn without edge crossings on a torus.

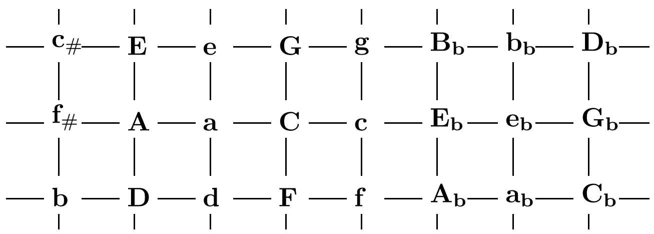


Figure 5: Regional kinship graph Γ_{Weber} according to Gottfried Weber

Lerdahl defines a metric $\Delta : Reg \times Reg \to [0, \infty)$ which quantitatively specifies and extends the kinship relation $Kin \subset Reg \times Reg$ to all pairs of regions. To all edges $(\mathbf{R}, \mathbf{R}') \in Kin$ he attributes the same distance value

$$\Delta(\mathbf{R}, \mathbf{R}') = 7.$$

Besides these two more types of regional kinship are selected to which larger *direct* distance values are attributed, namely:

1. Kinship to the *Leittonwechsel*-regions

$$\Delta(\mathbf{R}, \mathbf{R}') = 9 \quad \text{for all} \quad (\mathbf{R}, \mathbf{R}') \in Kin_L := \{(\mathbf{C},\mathbf{e}),(\mathbf{G},\mathbf{b}),...,(\mathbf{F},\mathbf{a})\}$$

2. Kinship to the *Supertonic* regions

$$\Delta(\mathbf{R}, \mathbf{R}') = 10 \quad \text{for all} \quad (\mathbf{R}, \mathbf{R}') \in Kin_S := \{(\mathbf{C},\mathbf{d}),(\mathbf{G},\mathbf{a}),...,(\mathbf{F},\mathbf{g})\}$$

In terms of graph theory we may say, that the extended graph

$$\Gamma^*_{Weber} = (Regions, Kin \sqcup Kin_L \sqcup Kin_S)$$

is colored by the distance labels $\Delta = 7, 9, 10$. Figure 6 shows two local subgraphs namely the *stars* of the nodes **C** and **a**, while figure 7 (upper part) displays the extended graph without edge labels. NB: This graph is not drawable on a torus without edge crossings.

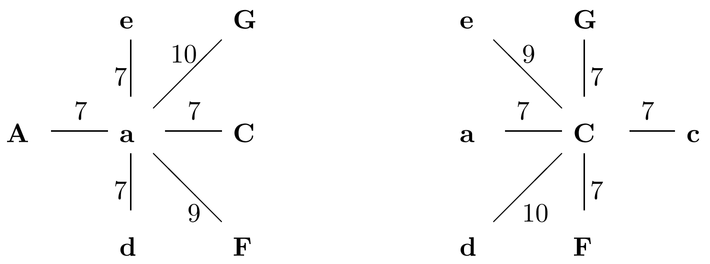


Figure 6: Stars of the nodes **C** and **a** within the extended regional kinship graph Γ^*_{Weber} with direct distance labels $\Delta = 7, 9, 10$.

On other region pairs $(\mathbf{R_1}, \mathbf{R_2})$ the metric Δ is defined in terms of *indirect* kinship. To each sequence $\sigma = (\mathbf{R_1}, ..., \mathbf{R_{n+1}})$ one can attribute its length $length_\Delta(\sigma) = \sum_{k=1}^{n} \Delta(\mathbf{R_k}, \mathbf{R_{k+1}})$ and defines

$$\Delta(\mathbf{R}, \mathbf{R}') = \min_{\sigma=(\mathbf{R},...,\mathbf{R}')} length_\Delta(\sigma)$$

The lower part of Figure 7 displays the distances $\Delta(\mathbf{C}, \mathbf{R})$ from the fixed C-major Region to all 24 regions. We obtain one deviation from Lehrdahl (2001) p. 69, namely $\Delta(\mathbf{C}, \mathbf{G_b}) = 28$ instead of $\Delta(\mathbf{C}, \mathbf{F_\#}) = 30$, which is not minimal.

5.3 Hybrid Space: Hierarchy versus Shortest Path

How to combine the intra-regional and the inter-regional distances? A solution is to just identify the regional loci in Weber Space with the corresponding regional centers in the disjoint union of all 24 regional spaces:

$$H_{168} := \bigsqcup_{k=0,...,11} Maj(\mathbf{k}) \quad \sqcup \quad \bigsqcup_{k=0,...,11} Min(\mathbf{k}).$$

If we dont specify the mode of a regional locus $\mathbf{R}$ we write $Mode(\mathbf{R})$ for the corresponding region and $\mathbb{I}/\mathbf{R}$ for its center. The intra-regional distances $\delta_{Mode(\mathbf{R})}$ together with the inter-regional distance Δ partially define a metric which can be canonically extended to a metric $\delta_{strict} : H_{168} \to H_{168}$. Consider two loci $\mathrm{X}_1/\mathbf{R_1}, \mathrm{X}_2/\mathbf{R_2}$ in different regions. Their strict distance is then

$$\begin{aligned}&\delta_{strict}(\mathrm{X}_1/\mathbf{R_1}, \mathrm{X}_2/\mathbf{R_2}) \\ &\quad = \delta_{R_1}(\mathrm{X}_1/\mathbf{R_1}, \mathbb{I}/\mathbf{R_1}) + \boldsymbol{\Delta}(\mathbf{R_1}, \mathbf{R_2}) + \delta_{\mathbf{R_2}}(\mathbb{I}/\mathbf{R_2}, \mathrm{X}_2/\mathbf{R_2}).\end{aligned}$$

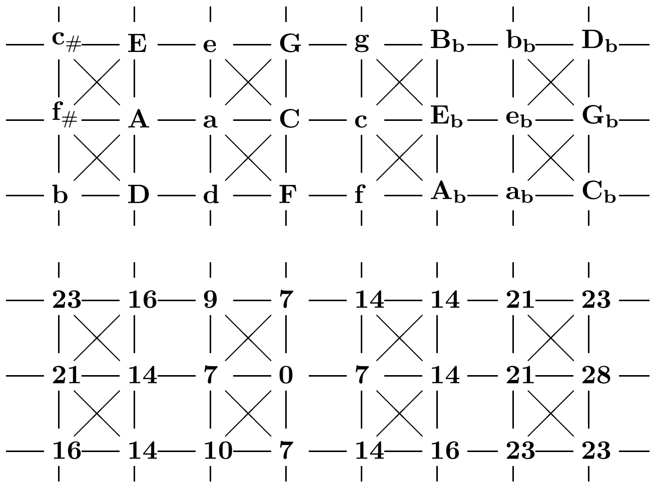


Figure 7: Extended regional kinship graph Γ^*_{Weber} (above) and corresponding distances $\Delta(\mathbf{C}, \mathbf{R})$ for varying region $\mathbf{R}$ (below)

However, what Lerdahl actually proposes is not this strict hierarchy, but a weakened version of it, where each locus is directly connected with six regional centers instead of just its own one. These regions are called *pivot regions.* Figure 8) displays the *chart* of the six pivot regions for the loci in the C-major region Maj($\mathbf{C}$). The same six regions serve also as pivot regions for loci of the a-Minor region Min($\mathbf{a}$). The proposal resembles the quasi-hierarchical organisiation of public transport. Local trains (or busses) move between peripherical loci and several regional centers while inter-regional trains connect regional centers but do not stop at peripherical loci. The pivot region chart of an arbitrary major region $\mathbf{R} = \mathbf{k}_+$ is the set

$$Pivot(\mathbf{k}_+) = \{(\mathbf{k+2})_-, (\mathbf{k+1})_-, (\mathbf{k+4})_-, (\mathbf{k-1})_+, \mathbf{k}_+, (\mathbf{k+1})_+\}.$$

The same pivot region chart is associated with the minor region $(\mathbf{k+3})_-$ relative to $\mathbf{R}$, i.e. we have $Pivot(\mathbf{k}_+) = Pivot((\mathbf{k+3})\_)$. The local pivot distances

$$\begin{aligned} \delta_{pivot,k_+} &: \quad Maj(\mathbf{k}) \times Pivot(\mathbf{k}_+) \to [0,\infty) \\ \delta_{pivot,k_-} &: \quad Min(\mathbf{k}) \times Pivot(\mathbf{k}_-) \to [0,\infty) \end{aligned}$$

are specified in table 5.3.

This map is extended to a map $\tau_{168} : H_{168} \times H_{168} \to [0,\infty)$ which is calculated as follows. For loci in the same region $\mathbf{R}$ τ_{168} coinsides with the intra-regional distance δ_R. For loci in different regions $\tau_{168}(\mathrm{X}_1/\mathbf{R_1}, \mathrm{X}_2/\mathbf{R_2})$ is the minimum length of the 36 indirect possible pathways between these loci via their pivot regions, i.e.

$$\begin{aligned} &\tau_{168}(\mathrm{X}_1/\mathbf{R_1}, \mathrm{X}_2/\mathbf{R_2}) \\ &\qquad = \min_{S,T} \left(\delta_{pivot,S}(\mathrm{X}_1/\mathbf{R_1}, \mathbf{S}) + \Delta(\mathbf{S}, \mathbf{T}) + \delta_{pivot,R_2}(\mathrm{X}_2/\mathbf{R_2}, \mathbf{T})\right), \end{aligned}$$

where $S \in Pivot(\mathbf{R_1}), T \in Pivot(\mathbf{R_2})$ vary through the pivot regions of $\mathbf{R_1}$ and $\mathbf{R_2}$ respectively. Lerdahl calls τ_{168} *chordal/regional distance* without discussing the

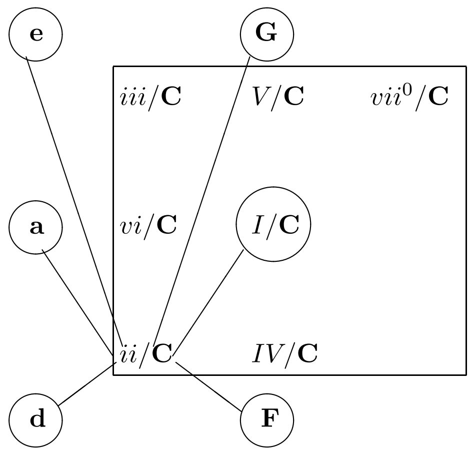


Figure 8: Chart of the six pivot regions for the loci in the C-major region. The edges represent the kinship relations of the second degree ii/**C**
to its pivot region centers

fact that it does *not* satisfy the triangle inequality. It is actually a *para-pseudo-distance* according to our terminology in subsection 1.3. The following example schows a violation of the triangle inequality between the regional centers I/**G** and i/**d**. On the hand we have the

$$\tau_{168}(\mathrm{I}/\mathbf{G}, \mathrm{i}/\mathbf{d}) = \Delta(\mathbf{G}, \mathbf{d}) = 14.$$

On the other hand we have

$$\tau_{168}(\mathrm{I}/\mathbf{G}, \mathrm{ii}/\mathbf{C}) + \tau_{168}(\mathrm{ii}/\mathbf{C}, \mathrm{i}/\mathbf{d}) = 7 + 2 = 9.$$

How to music-theoretically interpret this violation? As long as the notion of distance serves just an illustrative metaphor there is not need to insist in the model of a metric space. In terms of the public transport metaphor one could easily switch from a shortest-path-principle to a cheapest-ticket-principle, where a violation of the triangle-inequality is typically compensated by a higher comfort of travel. Within Lerdahl's theoretical framework, however, the violation has to be regarded as counterintuitive, because he simultaneously bilances the length (or cost) of pathways at several levels of reduction. It is as if a passenger in a local train - who manages to sleep with one eye closed at local stops - would be charged the high-comfort-price of an inter-regional train for the closed eye.

An alternative to the para-pseudo metrics τ_{168} is the pseudo-metrics δ_{168} : $H_{168} \times H_{168} \rightarrow [0, \infty)$, which minimizes the path lengths of all pathways between two loci in H_{168} along the edges defined by all intra-regional, pivotal and inter-regional distances. But the Weber space is not isometrically embedded into (H_{168}, δ_{168}).

Table 3: Local pivot distances of chordal loci in the C-major and the a-minor region to their common pivot regions

δ_{pivot}	i/**d**	i/**a**	i/**e**	I/**F**	I/**C**	I/**G**
I/**C**	10	7	9	7	0	7
ii/**C**	2	5	10	9	8	7
iii/**C**	10	5	2	10	7	9
IV/**C**	9	7	10	2	5	10
V/**C**	7	8	9	10	5	2
vi/**C**	7	0	7	9	7	10
vii^o/**C**	9	8	7	7	8	9
i/**a**	7	0	7	9	7	10
ii^o/**a**	9	8	7	7	8	9
III/**a**	10	7	9	7	0	7
iv/**a**	2	5	10	9	8	7
V/**a**	11	6	4	11	9	11
VI/**a**	9	7	10	2	5	10
#vii^o/**a**	8	9	11	11	8	6

5.4 Reduced Chordal/Regional Space

Theoretical issues aside, one may experimentally apply τ_{168} or δ_{168} to calculate optimal pathways through chordal/regional space H_{168} at just one level of analysis. The considerations of this subsection are useful in order to reduce the amount of calculation in such experiments. We show that the transition value maps τ_{168} and δ_{168} both contain a certain amount of redundancy which allows to reduce the number of loci to 108 instead of 168. The following definitions introduce suitable notions for the formulation of such redundancies.

Definition 1 *Consider a transition value map $\tau : H \times H \to [0, \infty)$. Two harmonic loci h_1 and h_2 are called extensionally indistinguishable with respect to the transition value map τ, if $\tau(h_1, h_2) = \tau(h_2, h_1) = 0$. The are called intensionally indistinguishable, if $\tau(h_1, g) = \tau(h_2, g)$ and $\tau(g, h_1) = \tau(g, h_2)$ for all loci $g \in H$, different from h_1 and h_2. Finally, they are called indistinguishable, if they are both, extensionally and intensionally indistinguishable.*

The following proposition provides a construction for a reduction of a given space H with respect to the intensional indistinguishabilities of a transition value map τ.

Proposition 1 *For symmetric transition value maps $\tau : H \times H \to R$, with $\tau(h_1, h_2) = \tau(h_1, h_2)$ for all $h_1, h_2 \in H$ intensional indistinguishability is an equivalence relation and τ induces a transition value map on the equivalence classes.*

Proof: Reflexivity and Symmetry are evident. As to the transitivity, let h_1 be intensionally indistinguishable from h_2 and let h_2 be intensionally indistinguishable from h_3. If $g \in H$ is any locus different from h_2 we have $\tau(g, h_1) = \tau(g, h_2) = \tau(g, h_3)$ as well as $\tau(h_1, g) = \tau(h_2, g) = \tau(h_3, g)$. In the case of $g = h_2$ we have $\tau(h_2, h_1) = \tau(h_3, h_1) = \tau(h_1, h_3) = \tau(h_2, h_3)$. The last statement follows from the definition of intensional indistinguishability. □

Now we consider our particular cases (H_{168}, τ_{168}) and (H_{168}, δ_{168}). If two loci are intensionally indistinguishable with respect to all the partial intra-regional, pivotal and inter-regional distances then this property is inherited by both τ_{168} and δ_{168} according to their construction. As we will see in subsection 5.6 all these partial distances are calculated on the basis of diatonic stratifications of triads. As certain loci in modally relative regions have the same stratifications and they are intensionally indistinguishable. In the case of C-major and a-minor regions we have:

$$\mathrm{I}/\mathbf{C} \sim \mathrm{III}/\mathbf{a},\ \mathrm{ii}/\mathbf{C} \sim \mathrm{iv}/\mathbf{a},\ \mathrm{IV}/\mathbf{C} \sim \mathrm{VI}/\mathbf{a},\ \mathrm{vi}/\mathbf{C} \sim \mathrm{i}/\mathbf{a},\ \mathrm{vii}^{o}/\mathbf{C} \sim \mathrm{ii}^{o}/\mathbf{a}$$

Each of the twelve pairs of modally relative regions consists of 9 equivalence classes (c.f. figure 9). The resulting *reduced chordal/regional space* consists of 108 loci and is denoted by H_{108}. It can be equipped either with a para-distance τ_{108} (inherited from τ_{168}) or with a distance δ_{108} (inherited from δ_{168}).

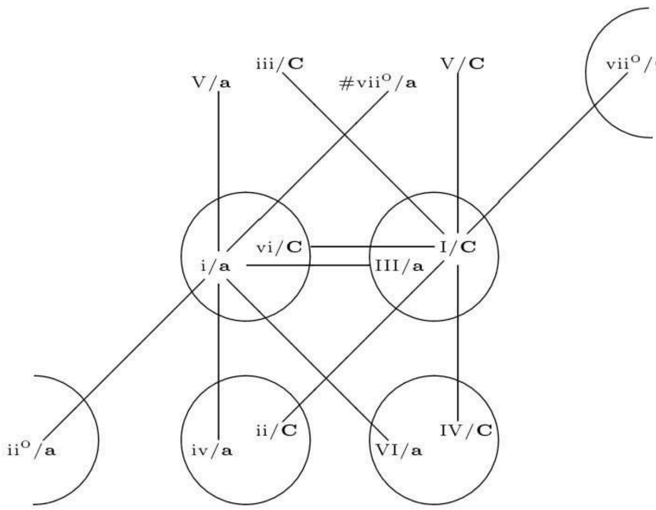


Figure 9: Classes of intensionally indistinguishable loci in modally relative regions

5.5 Lerdahl versus Lerdahl: The *Faith*-motive

The following example shows that Lerdahl did not strongly apply his theoretical principles from the chapter 2 of his book to the examples discussed in chapter 3 of his book. For the discussion of the harmonic pathway analysis of the *Faith motive* from Wagner's *Parsifal* (bars 45 - 55) we may assume that the space $CHORDS = TRIADS$ consists just of the 36 major, minor and diminished triads. The analysis starts from a previously obtained sequence $(X_0, X_1, ..., X_n)$ of triads. As each triad is associated with several harmonic loci, there is an associated family of combinatorially possible pathways to each chord sequence $(X_0, X_1, ..., X_n)$ among which a shortest one has to be chosen. Each major chord signifies 6 harmonic loci. If I/**I** is the center of a major region, there are six harmonic loci signified by one and the same major triad, e.g. with $\{c, e, g\}$:

$$\{\mathrm{I}/\mathbf{I}, \mathrm{V}/\mathbf{IV}, \mathrm{IV}/\mathbf{V}, \mathrm{III}/\mathbf{vi}, \mathrm{VI}/\mathbf{iii}, \mathrm{V}/\mathbf{iv}\}.$$

With respect to the same regional center there are five harmonic loci signified by one and the same minor triad, e.g. with $\{a, c, e\}$:

$$\{\mathrm{i}/\mathbf{vi}, \mathrm{vi}/\mathbf{I}, \mathrm{ii}/\mathbf{V}, \mathrm{iii}/\mathbf{IV}, \mathrm{iv}/\mathbf{iii}\}.$$

Finally, there are three harmonic loci signified by each diminished triad, e.g. with $\{b, d, f\}$:

$$\{\mathrm{vii}^o/\mathbf{I}, \mathrm{ii}^o/\mathbf{vi}, \#\mathrm{vii}^o/\mathbf{i}\}.$$

We may paraphrase these facts within our framework by attributing a boolean Riemann Logic $RL : TRIADS \times H_{108} \to \{0, 1\}$ sending each possible signification $(X \triangleright L)$ to 1 if they are associated to one another like described above and to 0 otherwise. If $\nu_{maj}, \nu_{min}, \nu_{dim}$ denote the total numbers of occurances of major, minor and diminished triads within the sequence $(X_0, X_1, ..., X_n)$ the number of possible harmonic pathways equals $6^{\nu_{maj}} \cdot 5^{\nu_{min}} \cdot 3^{\nu_{dim}}$.

Lerdahl reduces the *Faith motive* passage to the following sequence of 12 triads (7 major, 4 minor and 1 diminished):

$$(A_b+, E_b+, C_b+, G_b+, D+, F_\#-, C_\#-, F_b+, A_b-, F^o, B_b+, E_b-)$$

The resulting number of possible pathway-analyses is $6^7 \cdot 5^4 \cdot 3 = 524.880.000$. Lerdahl proposes the following pathway out of these:

$$\mathrm{I}/\mathbf{A}_b \rightsquigarrow \mathrm{V}/\mathbf{A}_b \rightsquigarrow \mathrm{I}/\mathbf{B} \rightsquigarrow \mathrm{V}/\mathbf{B} \rightsquigarrow \mathrm{I}/\mathbf{D} \rightsquigarrow \mathrm{iii}/\mathbf{D} \implies \mathrm{iv}/\mathbf{c}_\# \rightsquigarrow$$

$$\rightsquigarrow \mathrm{i}/\mathbf{c}_\# \rightsquigarrow \mathrm{III}/\mathbf{c}_\# \rightsquigarrow \mathrm{iv}/\mathbf{e_b} \rightsquigarrow \mathrm{ii_o}/\mathbf{e_b} \rightsquigarrow \mathrm{V}/\mathbf{e_b} \rightsquigarrow \mathrm{i}/\mathbf{e_b}$$

In order to make a clear balance, we first make the following observation: The devitions between the para-pseudo-distance τ_{168} and the pseudo-distance δ_{168} are not relevant in this particular example, as there are not transitions in the chord-sequence where such deviations occur. After reduction to the distance δ_{108} the number of possible pathways reduces to $4^7 \cdot 3^4 \cdot 2^1 = 2.654.208$. Lerdahl's analysis leads to a path length 99. There are 4676 paths with the same length. Furthermore, there are 8327 paths with lengths shorter than 99 and thus representing better analyses in sense of Lerdahl's theoretical approach. There are 22 analyses representing shortest possible paths of length 89:

$$\mathrm{IV}/\mathbf{E_b} \rightsquigarrow \mathrm{I}/\mathbf{E_b} \rightsquigarrow \mathrm{IV}/\mathbf{G_b} \rightsquigarrow \mathrm{I}/\mathbf{G_b} \rightsquigarrow \mathrm{IV}/\mathbf{A} \rightsquigarrow$$

$$\left[\begin{array}{lllllll}
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{vi}/\mathbf{E} & \rightsquigarrow & \text{I}/\mathbf{E} & \rightsquigarrow & \text{vi}/\mathbf{B} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{vi}/\mathbf{E} & \rightsquigarrow & \text{I}/\mathbf{E} & \rightsquigarrow & \text{ii}/\mathbf{G_b} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{vi}/\mathbf{E} & \rightsquigarrow & \text{I}/\mathbf{E} & \rightsquigarrow & \text{iii}/\mathbf{E} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{vi}/\mathbf{E} & \rightsquigarrow & \text{IV}/\mathbf{B} & \rightsquigarrow & \text{vi}/\mathbf{B} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{vi}/\mathbf{E} & \rightsquigarrow & \text{IV}/\mathbf{B} & \rightsquigarrow & \text{ii}/\mathbf{G_b} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{ii}/\mathbf{B} & \rightsquigarrow & \text{IV}/\mathbf{B} & \rightsquigarrow & \text{vi}/\mathbf{B} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{ii}/\mathbf{B} & \rightsquigarrow & \text{IV}/\mathbf{B} & \rightsquigarrow & \text{ii}/\mathbf{G_b} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{iii}/\mathbf{A} & \rightsquigarrow & \text{I}/\mathbf{E} & \rightsquigarrow & \text{vi}/\mathbf{B} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{iii}/\mathbf{A} & \rightsquigarrow & \text{I}/\mathbf{E} & \rightsquigarrow & \text{ii}/\mathbf{G_b} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{iii}/\mathbf{A} & \rightsquigarrow & \text{I}/\mathbf{E} & \rightsquigarrow & \text{iii}/\mathbf{E} \\
\hline
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{iii}/\mathbf{A} & \rightsquigarrow & \text{V}/\mathbf{A} & \rightsquigarrow & \text{vi}/\mathbf{B} \\
\hline
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{iii}/\mathbf{A} & \rightsquigarrow & \text{V}/\mathbf{A} & \rightsquigarrow & \text{ii}/\mathbf{G_b} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{iii}/\mathbf{A} & \rightsquigarrow & \text{V}/\mathbf{A} & \rightsquigarrow & \text{iii}/\mathbf{E} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{iii}/\mathbf{A} & \rightsquigarrow & \text{IV}/\mathbf{B} & \rightsquigarrow & \text{vi}/\mathbf{B} \\
\text{vi}/\mathbf{A} & \rightsquigarrow & \text{iii}/\mathbf{A} & \rightsquigarrow & \text{IV}/\mathbf{B} & \rightsquigarrow & \text{ii}/\mathbf{G_b} \\
\text{ii}/\mathbf{E} & \rightsquigarrow & \text{vi}/\mathbf{E} & \rightsquigarrow & \text{I}/\mathbf{E} & \rightsquigarrow & \text{vi}/\mathbf{B} \\
\text{ii}/\mathbf{E} & \rightsquigarrow & \text{vi}/\mathbf{E} & \rightsquigarrow & \text{I}/\mathbf{E} & \rightsquigarrow & \text{ii}/\mathbf{G_b} \\
\text{ii}/\mathbf{E} & \rightsquigarrow & \text{vi}/\mathbf{E} & \rightsquigarrow & \text{I}/\mathbf{E} & \rightsquigarrow & \text{iii}/\mathbf{E} \\
\text{ii}/\mathbf{E} & \rightsquigarrow & \text{vi}/\mathbf{E} & \rightsquigarrow & \text{IV}/\mathbf{B} & \rightsquigarrow & \text{vi}/\mathbf{B} \\
\text{ii}/\mathbf{E} & \rightsquigarrow & \text{vi}/\mathbf{E} & \rightsquigarrow & \text{IV}/\mathbf{B} & \rightsquigarrow & \text{ii}/\mathbf{G_b} \\
\text{ii}/\mathbf{E} & \rightsquigarrow & \text{ii}/\mathbf{B} & \rightsquigarrow & \text{IV}/\mathbf{B} & \rightsquigarrow & \text{vi}/\mathbf{B} \\
\text{ii}/\mathbf{E} & \rightsquigarrow & \text{ii}/\mathbf{B} & \rightsquigarrow & \text{IV}/\mathbf{B} & \rightsquigarrow & \text{ii}/\mathbf{G_b}
\end{array}\right]$$

$$\rightsquigarrow \text{vii}^{o}/\mathbf{G_b} \rightsquigarrow \text{III}/\mathbf{G_b} \rightsquigarrow \text{vi}/\mathbf{G_b}$$

All of the 22 pathways start with the same 5 loci and end with the same 3 loci and differ only in the choice of the 4 loci inbetween. The authors were surprised by the observation, that Cohn (2003) presented exactly one of those shortest analyses as a >Lerdahl-Style-Analysis<, namely No. 11 (enclosed between horizontal lines) as a point of departure for his further analysis. The shortest analyses support a plagal interpretation of the opening rising fifths. Lerdahl's analysis is in better accordance with David Lewin's observation, that the regional centers of this passage form a *Klingsor motive* (c.f. Lewin (1987)).

5.6 Diatonic Triad Stratifications

In this last subsection we recapitulate Lerdahl's definition of the elementary intra-regional, inter-regional and local pivot-distances. We introduce our own terminology and eliminate a technical problem.

A *diatonic locus* $(\tau_{type}(k), Dia(n))$ is a triad with(in) a diatonic collection, where *type* $\in \{Maj, min, dim\}$ and $n, k \in \mathbb{F}$ such that $|(\tau_{type}(k)| \subset |Dia(n)|$ ($|S|$ denotes the *set* of the elements occouring in a sequence S). As an example consider $I_C = (\tau_{Maj}(0), Dia(0))$, which denotes the C major triad within the 'C-major' diatonic collection.[9]

An *altered diatonic locus* is a triad within an altered diatonic scale, i.e. a pair $(\tau_{type}(k), Dia(n)_\Sigma)$, such that $|(\tau_{type}(k)| \subset |Dia(n)_\Sigma|$. As an example consider

9 The label 'C-major' should not include a modal connotation when it is applied to a diatonic collection.

$V_a = (\tau_{Maj}(4), Dia(0)_{(0,0,1,0,0,0,0)})$ denoting the E major triad within the corresponding alteration of the C-major/a-minor diatonic collection. The sets of all diatonic and altered diatonic loci are denoted by $\mathcal{L}_{dia}$ and $\mathcal{L}_{alt}$ respectively. They can ce studied as harmonic configuration spaces in their own right equipped with a harmonic tensor *ht* from which Lerdahl derives the desired elementary values.

The *Lerdahl stratification* of an (altered) diatonic locus (τ, D) is the chain of three set inclusions $Strat(\tau, D) := \quad \{\tau[1]\} \subset \{\tau[1], \tau[2]\} \subset |\tau| \subset |D|$, where $\tau[1]$ and $\tau[2]$ denote the first and the second element of the sequence τ respectively. Given two diatonic loci (τ_1, D_1) and (τ_2, D_2) we define the *difference vector*

$$Diff((\tau_1, D_1), (\tau_2, D_2)) := \begin{pmatrix} \sharp(\{\tau_2[1]\}\backslash\{\tau_1[1]\}) \\ \sharp(\{\tau_2[1], \tau_2[2]\}\backslash\{\tau_1[1], \tau_1[2]\}) \\ \sharp(|\tau_2|\backslash|\tau_1|) \\ \sharp(|D_2|\backslash|D_1|) \end{pmatrix}$$

The difference vector counts the number differences between the two diatonic loci at all four levels of the corresponding Lerdahl stratifications separately. In order to give the possibility to give different weight to the four levels we introduce a fixed user defined *stratification profile* $p_{strat} = (p_1, p_2, p_3, p_4)$ with nonnegative real entries p_1, p_2, p_3, p_4 and define the (weigthed) *difference sum*

$$diff((\tau_1, D_1), (\tau_2, D_2)) := \langle p_{strat}, Diff((\tau_1, D_1), (\tau_2, D_2))\rangle$$

The default setting for the stratification profile is $p_{strat} = (1, 1, 1, 1)$ (c.f. Lehrdahl (2001), p. 55, 60).

Furthermore Lerdahl takes the diatonic distances between the triads and the distances between the diatonic collections into account. In order to weight their contribution to the harmonic tensor *ht* we fix another pair (q_1, q_2) of nonnegative real weights and define:

$$\begin{aligned} &dist((\tau_{type1}(k_1), Dia(n_1)_{\Sigma_1}), (\tau_{type_2}(k_2), Dia(n_2)_{\Sigma_2}) \\ &\qquad := q_1 \cdot d_{dia}(k_1, k2) + q_2 \cdot d(n1, n2) \end{aligned}$$

Again, the default setting for the distance weigths is $q_{dist} = (q_1, q_2) = (1, 1)$. The *harmonic tension* between two (altered) diatonic loci $X_1 = (\tau_1, D_1)$ and $X_2 = (\tau_2, D_2)$ is calculated by adding their distance to the difference sum:

$$ht(X_1 \rightsquigarrow X_2) \quad = \quad \delta(X_1, X_2) \quad := \quad dist(X_1, X_2) + diff(X_1, X_2)$$

Proposition 2 *The harmonic tension $ht = \delta$ is a pseudo-distance on the space $\mathcal{L}_{alt}$ of altered diatonic loci. It is a distance, if the two profile parameters p_3 and p_4 do not vanish.*

Proof: We first check the triangle inequality $\delta(X_1, X_2) + \delta(X_2, X_3) \geqslant \delta(X_1, X_3)$ for all $X_1, X_2, X_3 \in \mathcal{L}_{alt}$. We show that it independently holds for all four components of the difference vector as well as for the distances d and d_{dia}. As to the differences suppose one is given three finite sets A, B, C. One has

$$\begin{aligned} card(B\backslash A) &= card(B\backslash(A \cup C)) &+& \; card(B \cup (C\backslash A)) \\ card(C\backslash B) &= card(C\backslash(B \cup A)) &+& \; card(C \cup (A\backslash B)) \\ card(C\backslash A) &= card(C\backslash(A \cup B)) &+& \; card(C \cup (B\backslash A)) \end{aligned}$$

The triangle inequality immediately follows from $C \cup (B \backslash A) = B \cup (C \backslash A)$. The triangle inequality for d is evident because it is a metric on $\mathbb{Z}$. Its additive invariance implies that d_{dia} is a pseudometric (to each triangle in the diatonic classes $\mathbb{Z}_7$ one finds a triangle in $\mathbb{Z}$ representing the diatonic distances.)

The symmetry condition $\delta(X_1, X_2) = \delta(X_2, X_1)$ for all X_1 and X_2 holds separately for all components of the difference vector due to the fact that the cardinalities $1, 2, 3, 7$ of the stratification levels are fixed and hence the same for X_1 and X_2. For any two finite sets A and B of equal cardinality one always has $card(A \backslash B) = card(B \backslash A)$. The symmetry of d and d_{dia} follows from the fact, that they are a metric and a pseudometric respectively. So far we have shown that δ is a pseudometric on $\mathcal{L}_{alt}$.

As to the metric, suppose $X_1 = (\tau_1, D_1)$ and $X_2 = (\tau_2, D_2)$ do not coincide. In the case $\tau_1 \neq \tau_2$ we have $\delta(X_1, X_2) > p_3$ and in the case $D_1 \neq D_2$ we have $\delta(X_1, X_2) > p_2$.

Remark 7 *Lerdahl's hierarchical* tonal pitch space *model is not explicitely based on the space $\mathbb{F}$ of note names, but — according to his own explanations — on the space $\mathbb{H}_{oct}$ of the twelve octave classes of pitch height. He studies twelve diatonic collections $dia^*(k) := k + \{0, 2, 4, 5, 7, 9, 11\} \subset \mathbb{H}_{oct}$ for $k = 0, ..., 11$ as 7-elemented sets of pitch classes. However, close examination shows that the diatonic distance d_{dia} between roots of triads cannnot be properly defined in $\mathbb{H}_{oct} \cong \mathbb{Z}_{12}$. In other words, the* chord distance rule *(c.f. p. 60) is not properly defined and Lerdahl's argument on page 63, denying a »fault not of the the rule« is not correct. Nevertheless the desired intra-regional, inter-regional and local pivot distances can be properly defined by choosing small subdomains of $\mathbb{F}$ without enharmonic ambiguities of the diatonic distance.*

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